DERIVATION OF EQUATIONS OF MOTION OF A SLIGHTLY RAREFIED GAS AROUND HIGHLY HEATED BODIES FROM BOLTZMANN'S EQUATION

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The motion of a slightly rarefied gas (K ≪ 1, where K is the Knudsen number) around highly heated bodies is examined. On the assumption that the characteristic macroscopic velocity of gas motion generated during contact with a highly heated body is on the order of or much greater than the velocity of the impinging stream, the corresponding hydrodynamic equations are derived from Boltzmann's equation by Hilbert's method [1]. A qualitative study is made of the region of applicability of the equations obtained. A class of flows of a continuous medium in which the characteristic change in enthalpy is much larger than the characteristic kinetic energy was studied in [2]. The Navier–Stokes equations with boundary conditions of adhesion proved to be inadequate for a description of these flows since it was already necessary in the first basic approximation to take into account part of the Barnett terms and slippage. The authors of [2] suggest using simplified Barnett equations with the condition of creep, with the Barnett terms being on the same order as the inertial and Navier–Stokes terms. On the other hand, it is known that the Barnett equations are derived on the assumption that the additional terms are small in comparison with the Navier–Stokes and Eulerian terms. This makes it desirable to obtain equations describing this class of flows directly from Boltzmann's equation.

1. As shown in [3], the motion of a gas around highly heated bodies must be classified as a function of the parameter

$$w = u_\infty / u_0 \tag{1.1}$$

where u_0 is the characteristic velocity of the macroscopic gas motion produced during contact with the highly heated body and u_∞ is the velocity of the impinging stream.

Simple estimates show that $u_0 \sim \varepsilon\nu/L$, where ν is the kinematic coefficient of viscosity, L is the characteristic linear dimension of the body around which the flow occurs, $\varepsilon \sim \Delta T/T \leq o\ (1)$, and ΔT is the characteristic temperature drop.

When $w \gg 1$ the gas motion is described by the usual Navier–Stokes equations [3].

The flows of a continuous medium when $w \sim 1$ and $w \ll 1$ are of interest. In this case the characteristic velocity of the motion proves to be on the order of u_0, and the Reynolds number can be estimated in the following way:

$$\mathrm{Re} = u_0 L/\nu \sim \varepsilon \tag{1.2}$$

In addition, $\varepsilon \gg M_\infty{}^2$, where M_∞ is the Mach number of the impinging stream. The Boltzmann's equation for this class of flows has the usual form [4]

$$\mathrm{K}\,(\partial f/\partial t + \xi_i\, \partial f/\partial x_i) = J\,(f, f) \tag{1.3}$$

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We will seek a solution for Eq. (1.3) in the form

$$f = {}^{(0)} + Kf^{(1)} + K^2 f^{(2)} + \ldots \tag{1.4}$$

Substituting (1.4) into (1.3) and collecting terms with the same powers of K we obtain the recurrent system of equations

$$\begin{aligned} &J(f^{(0)}, f^{(0)}) = 0 \\ &2J(f^{(1)}, f^{(0)}) = \frac{\partial f^{(0)}}{\partial t} + \xi_i \frac{\partial f^{(0)}}{\partial x_i} \\ &2J(f^{(2)}, f^{(0)}) = \frac{\partial f^{(1)}}{\partial t} + \xi_i \frac{\partial f^{(1)}}{\partial x_i} - J(f^{(1)}, f^{(1)}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{1.5}$$

We determine the first five moments of the distribution function in the following way:

$$\begin{aligned} \rho &= \rho^{(0)} + K\rho^{(1)} + K^2\rho^{(2)} + \ldots \\ u_i &= Ku_i^{(1)} + K^2 u_i^{(2)} + \ldots \\ p &= p^{(0)} + Kp^{(1)} + K^2 p^{(2)} + \ldots \end{aligned} \tag{1.6}$$

The distribution function of the null approximation is locally Maxwellian, with

$$\rho^{(0)} = \int f^{(0)}\, d\xi, \qquad \rho^{(0)} u_i^{(0)} = \int \xi_i f^{(0)}\, d\xi = 0, \qquad 3p^{(0)} = \int \xi^2 f^{(0)}\, d\xi \tag{1.7}$$

For solvability of the next integral equation of system (1.5) the following condition must be satisfied:

$$\begin{gathered} \int \left(\frac{\partial f^{(0)}}{\partial t} + \xi_i \frac{\partial f^{(0)}}{\partial x_i} \right) \psi_r\, d\xi = 0 \\ (r = 0, 1, 2, 3, 4; \quad \psi_0 = 1;\ \psi_i = \xi_i,\ i = 1, 2, 3;\ \psi_4 = \xi^2) \end{gathered} \tag{1.8}$$

From this, taking (1.7) into account, we obtain

$$\frac{\partial \rho^{(0)}}{\partial t} = 0, \qquad \frac{\partial p^{(0)}}{\partial t} = 0, \qquad \frac{\partial p^{(0)}}{\partial x_i} = 0 \tag{1.9}$$

We seek the solution of the integral equation determining $f^{(1)}$ using the well-known method of [5]

$$\begin{gathered} f^{(1)} = -\frac{1}{\rho^{(0)}} f^{(0)} (2RT^{(0)})^{1/2} A(\xi)\, \xi_i \frac{\partial \ln T^{(0)}}{\partial x_i} + \sum_{r=0}^{4} f^{(0)} \gamma_r^{(1)} \psi_r \\ \left(T^{(0)} = \frac{p^{(0)}}{\rho^{(0)} R} \right) \end{gathered} \tag{1.10}$$

where R is the gas constant and $\gamma_r^{(1)}$ are certain functions of x and t, r = 0, 1, 2, 3, 4.

In particular, for Maxwellian molecules

$$A(\xi) = \frac{3\mu^{(0)}}{2RT^{(0)}} \left(\xi^2 - \frac{5}{2} \right)$$

where $\mu^{(0)}$ is the coefficient of viscosity of the null approximation, so that $\mu^{(0)} \sim T^{(0)}$.

From the condition of solvability of the integral equation for the distribution function of the second approximation we obtain a system of equations which $\rho^{(1)}$, $u_i^{(1)}$, and $p^{(1)}$ must satisfy:

$$\begin{gathered} \frac{\partial \rho^{(1)}}{\partial t} + \frac{\partial \rho^{(0)} u_i^{(1)}}{\partial x_i} = 0, \qquad \frac{\partial u_i^{(1)}}{\partial t} + \frac{1}{\rho^{(0)}} \frac{\partial p^{(1)}}{\partial x_i} = 0, \\ \frac{\partial p^{(1)}}{\partial t} + \frac{5}{2} p^{(0)} \frac{\partial u_i^{(1)}}{\partial x_i} + \frac{\partial q_i^{(1)}}{\partial x_i} = 0 \end{gathered} \tag{1.11}$$

Here the heat-flux vector $q_i^{(1)}$ of the first approximation is calculated from the distribution function (1.10)

$$q_i^{(1)} = -\lambda^{(0)}\,\partial T^{(0)}/\partial x_i \tag{1.12}$$

where $\lambda^{(0)}$ is the coefficient of thermal conductivity of the null approximation.

The conditions of solvability of the integral equation for $f^{(3)}$ lead to the system of equations

$$\begin{gathered}\frac{\partial\rho^{(2)}}{\partial t}+\frac{\partial}{\partial x_i}(\rho^{(1)}u_i^{(1)}+\rho^{(0)}u_i^{(2)})=0\\ \rho^{(0)}\frac{\partial u_r^{(2)}}{\partial t}+\rho^{(0)}u_i^{(1)}\frac{\partial u_r^{(1)}}{\partial x_i}-\frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial p^{(1)}}{\partial x_r}+\frac{\partial p^{(2)}}{\partial x_r}+\frac{\partial p_{ri}^{(2)}}{\partial x_i}=0\\ \frac{\partial p^{(2)}}{\partial t}+\frac{\partial}{\partial x_i}(p^{(0)}u_i^{(2)}+p^{(1)}u_i^{(1)})+\frac{2}{3}p^{(0)}\frac{\partial u_i^{(2)}}{\partial x_i}+\frac{2}{3}p^{(1)}\frac{\partial u_i^{(1)}}{\partial x_i}+\frac{2}{3}\frac{\partial q_i^{(2)}}{\partial x_i}=0\end{gathered} \tag{1.13}$$

The stress tensor $p_{ij}^{(2)}$ and the heat-flux vector $q_i^{(2)}$ of the second approximation are determined from the function $f^{(2)}$. The integral equation for $f^{(2)}$ is solved by the method of expansion by Sonine polynomials [6]. As a result we find

$$\begin{gathered}p_{ij}^{(2)}=-2\mu^{(0)}\left\langle\frac{\partial u_i^{(1)}}{\partial x_j}\right\rangle+K_1\frac{\mu^{(0)2}}{\rho^{(0)}T^{(0)}}\left\langle\frac{\partial^2 T^{(0)}}{\partial x_i\partial x_j}\right\rangle+K_2\frac{\mu^{(0)2}}{\rho^{(0)}T^{(0)}}\left\langle\frac{\partial T^{(0)}}{\partial x_i}\frac{\partial T^{(0)}}{\partial x_j}\right\rangle\\ q_i^{(2)}=-\lambda^{(1)}\frac{\partial T^{(0)}}{\partial x_i}-\lambda^{(0)}\frac{\partial T^{(1)}}{\partial x_i}\\ \langle A_{ij}\rangle={}^1/_2(A_{ij}+A_{ji})-{}^1/_3\delta_{ij}A_{KK}\end{gathered} \tag{1.14}$$

For Maxwellian molecules $\lambda^{(1)}=\lambda^{(0)}T^{(1)}/T^{(0)}$; K_1 and K_2 are constants.

In the particular case of stationary gas motion Eqs. (1.9), (1.11), and (1.13) are reduced to the system of equations

$$\begin{gathered}\frac{\partial p^{(0)}}{\partial x_i}=0,\qquad\frac{\partial\rho^{(0)}u_i^{(1)}}{\partial x_i}=0\\ \frac{5}{2}p^{(0)}\frac{\partial u_i^{(1)}}{\partial x_i}+\frac{\partial q_i^{(1)}}{\partial x_i}=0\\ \rho^{(0)}u_r^{(1)}+\frac{\partial u_i^{(1)}}{\partial x_r}+\frac{\partial p^{(2)}}{\partial x_i}+\frac{\partial p_{ir}^{(2)}}{\partial x_r}=0\end{gathered} \tag{1.15}$$

Since the slippage velocity caused by the temperature gradient at the wall is of a primary order of magnitude, as the boundary conditions at the wall for the system (1.15) one must take the condition of creep

$$u_{\tau}^{(1)}\big|_{n=0}=\beta\frac{\mu^{(0)}}{\rho^{(0)}T^{(0)}}\frac{\partial T^{(0)}}{\partial x_\tau}\bigg|_{n=0} \tag{1.16}$$

where n and τ are the normal and tangent to the surface, and β is a constant. The slippage velocity caused by the transverse velocity gradient and the temperature drop at the surface are of a much smaller order (according to the Knudsen number) than the primary velocity and temperature, respectively.

Then, just as in [2], one can show that the characteristic drop $P_{ij}^{(2)}$ ($P_{ij}^{(2)}=p_{ij}^{(2)}+p^{(2)}\delta_{ij}$) across the Knudsen layer is on the order of K^3, while the drop $q_i^{(1)}$ is on the order of K^2, and the characteristic variations of these values in the stream are on the order of K^2 and K, respectively. Therefore, the variation of $P_{ij}^{(2)}$ and $q_i^{(1)}$ in the Knudsen layer can be neglected.

2. The solutions of the system of equations (1.15) differ in general from the solutions of the simplified Barnett equations which are used in [2].

Let us examine, for example, the motion of an incompressible gas. In this case, because $\partial p^{(0)}/\partial x_i=0$, the terms in $p_{ij}^{(2)}$ connected with temperature gradients become equal to zero and the momentum equation of system (1.15) is reduced to the Navier–Stokes equation. Moreover, in solving this problem by the Chapman–Enskog method, i.e., using the Barnett equations, the terms connected with temperature gradients remain, and moreover, in a number of problems they are the largest terms.

The solutions of system (1.15) will differ from the solutions of the Barnett equations for the problem of the motion of a gas around an unevenly heated body with $\varepsilon \sim 1$ and large wall-temperature gradients, $\nabla T_W \sim 1$. If $\varepsilon \sim 1$ but $\nabla T_W \ll 1$, the solution of system (1.15) and of the Barnett equations leads to the same results.

The Hilbert and Chapman–Enskog methods also lead to the same results in a study of gas flows around uniformly heated bodies when the wall temperature differs little from the temperature of the impinging stream.

Thus the Hilbert method leads to the same results as the Chapman–Enskog method in the case of small temperature gradients.

It is usually assumed [4] that in the Hilbert method the number of boundary conditions in the corresponding hydrodynamic equations does not depend on the order of the approximation and is the same as for Euler's equations, whereas in the Chapman–Enskog method the number of these boundary conditions can increase with a growth in the number of approximations. Neither of these assumptions is confirmed in the study of the present problem. Because of the degeneracy of the problem the same number of boundary conditions are needed for the solution of the simplified Barnett equations as for the Navier–Stokes equations. On the other hand, Euler's equations in Hilbert's method do not give the flow any concrete definition. The same number of boundary conditions as in the Chapman–Enskog method are required in using the second approximation [system (1.15)].

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LITERATURE CITED

1. D. Hilbert, Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen, B. G. Teubner, Leipzig-Berlin (1912).
2. V. S. Galkin, M. N. Kogan, and O. G. Fridlender, "Some kinetic effects in flows of a continuous medium," Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, No. 3 (1970).
3. V. N. Zhigulev, "The problem of gas motion around highly heated bodies," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 4 (1972).
4. H. Grad, "Principles of the kinetic theory of gases," Handbuch der Physik, Vol. 12, Springer-Verlag, Berlin (1958).
5. S. Chapman and T. Cowling, Mathematical Theory of Nonuniform Gases, Cambridge Univ. Press (1970).
6. D. Burnett, "The distribution of molecular velocities and the mean motion in a nonuniform gas," Proc. London Math. Soc., Ser. 3, Vol. 40, Part 5 (1935).